Question B1 of Worksheet 3 asked:

> Explain carefully, using the definitions of a valid inference and of a tautology, why an inference in truth functional logic is valid just in case its associated conditional is a tautology.

Seeing as we are logicians, let's be precise about what we are being asked to do. "Just in case X" means "in the case that X and *only* in the case that X". i.e., It means you need to explain why:

- An inference in truth functional logic being valid means the associated conditional is a tautology; **and**
- An associated conditional being a tautology means the inference is valid

To start let's recall the definition of a valid inference and the definition of a tautology.

> **Definition: Validity** (from p. 45 of the notes) An inference form written in the language of truth functional logic is a VALID form iff there is no assignment of truth values to its atomic components which makes the premises true and the conclusion false.

> **Definition: Tautology** (from p. 39 of the notes) A truth functional compound is a tautology if and only if it takes the truth value true in all lines of its truth table.

So we are being asked to show these two things:

$V \Rightarrow T$: An inference has no assignments of truth values to its atomic components which makes the premises true and the conclusion false (i.e. being valid), means that its associated conditional takes the truth value 'true' in all lines of its truth table (is a tautology); and

$T \Rightarrow V$: The associated conditional taking the truth value 'true' in all lines of its truth table (i.e. being a tautology) means that the inference has no assignments of truth values to its atomic components which makes the premises true and the conclusion false (is valid)

Proof of $V \Rightarrow T$

Suppose we have a valid inference written schematically as follows:

$$\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline \therefore C \end{array} \tag{1}$$

P_1, P_2 etc are premises (and so stand for logical sentences like p, $p \to q$, $\neg q \land (r \lor \neg p)$ or whatever), and C is the conclusion. The associated conditional for this inference is

$$(P_1 \land P_2 \land \ldots \land P_n) \to C \tag{2}$$

By the definition of validity, if this inference is valid then in its truth table there is no assignment of truth values to the atomic components (the parts of P_1, P_2, etc.) that makes all of $P_1, P_2, \ldots, P_n$ true and C false. So there is **no line** in the truth table that looks like this:

p	q	$\ldots$	P_1	P_2	$\ldots$	P_n	C
T	F	$\ldots$	T	T	$\ldots$	T	F

Table 1: This line doesn't exist in inference (1)'s truth-table

Now consider the truth table for the associated conditional. It is a conditional (as the name says), which we could think of as $A \to C$, where

$$\underbrace{(P_1 \wedge P_2 \wedge \ldots \wedge P_n)}_{A} \to C$$
$$\to C$$

If there is no way for $P_1, P_2, \ldots, P_n$ to be true and C false, then in the associated conditional there is no line which looks like Table 2.

p	q	$\ldots$	A	C
T	F	$\ldots$	T	F

Table 2: This line doesn't exist in conditional (2)'s truth-table

But that line is the only line which can make $A \to C$ turn out false. Remember the truth table for the implication? What it means to say that there is no situation where A is true and C is false is that we never use the highlighted row of Table 3.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	F

Table 3: Truth-table for implication

But all the other rows make $p \to q$ come out true. So the truth table of $A \to C$ will have all rows coming out true, which is the definition of a tautology. □

Proof of $T \Rightarrow V$

Now start by supposing that $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \to C$ is a tautology.

Then there is no line in its truth table in which it comes out false. But by the same argument as above (looking at the truth table for the conditional) that must mean there is no situation where $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ is true and C false. The premises are in a big conjunction, and as you know an "and" is only true if all the parts of it are true. So this means that it is never the case that *all of* the P_1, P_2, ... are true, while C is false – no matter what truth values you give to the p's and q's which make up P_1, P_2, etc.

But that just is the definition of validity: there is no case where the premises are true and the conclusion is false. So the inference is valid. □