# Effective Field Theory

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**Abstract**

This essay is a review article on effective field theory. This is a technique in quantum field theory which allows for the construction of new theories from known Lagrangians; either 'top-down' (taking the known Lagrangian as the high-energy theory and deriving a low-energy effective theory) or 'bottom-up' (taking the known Lagrangian as the low-energy theory and incorporating it into an effective theory approximating an unknown high-energy theory). This article reviews the basics of effective field theory construction in each of these cases. The delivery is aimed at a Master's student with some knowledge of quantum field theory.

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# 1 Introduction

The effective theory approach is a technique in physics which exploits a simple but remarkably useful fact about nature: physical problems often come with distinct energy scales. The physical processes dominant at well-separated scales differ, and an effective theory focuses on the important physics in a manner which is conceptually clear and assists in calculations.

We are well acquainted with this idea, and frequently simplify physical problems by, for example, setting small masses to zero when they occur together with large masses. We are able to do this because of the experimental fact that most of the details of low energy physics are independent of the details of high energy physics. The history of physics tells us it had to be this way, as otherwise we would have had no success in the Newtonian domain without a proper understanding of particle physics. The aim of effective field theory (EFT) is to rigorously apply this idea to field theory, excluding as much of the detail of the high energy theory as possible when doing low energy physics. In order to do this we must ask two questions. First, which high energy features are important at low energies? Second, which low energy processes are *most* sensitive to the details of high energy physics? We can go some way toward answering this latter question even when we don't know the high energy theory; a remarkable feature of this approach.

There are many motivations for using EFTs:

1. They simplify calculations. This is useful when we don't want to waste time calculating the details of unimportant high energy dynamics, or when we simply don't know those details.

2. They can parametrise additions to known physics. By constructing an EFT extending our known theory, we can parametrise the impact on current observables due to new physics.

3. They clarify the link between old and new physical theories. For example, EFT shows us how the Fermi theory of the weak interaction arises from full electroweak theory and explains where and why it is accurate.

4. They explain why certain approximations are suspiciously accurate. EFT shows us how to organise calculations of results which depend on deep symmetries, such as the Quantum Hall effect, in a manner which explains why seemingly heavy-handed approximations don't compromise accuracy.

EFT has had a unique impact on quantum field theory, giving firm grounding to the physics of renormalisation. The natural endpoint of the thinking that led from the original *ad hoc* "cancellation of infinities" to the renormalisation group equations is EFT. It gives new meaning to the renormalisation group, and explains when and why renormalisable theories arise. EFT also rehabilitates non-renormalisable theories, by showing how and why they can give accurate results.

Taking effective field theory seriously means accepting that all our theories are at best effective theories of some higher, currently unknown, theories. They are all therefore only valid up to some energy level. This doesn't reduce their practical efficacy, as experiments can only probe finite energy ranges. Problems may occur if we can probe outside the range of our current theory's application. But this is simply a way of describing the business of high energy physics: our theories of physics are necessarily responses to or inspired by the experimental data available at a given energy range. We don't know the high energy theory of everything, so we design theories which fit energy scales accessible to us. We make predictions beyond the currently accessible energy range and then wait for experimental corroboration/falsification as a new generation of experiments come on-line.

Not all of the above will be discussed in depth here. My focus is the construction of EFTs, specifically in the context of a weakly interacting UV theory[1]. There are two cases: constructing an effective theory from a known high-energy theory and casting a known theory as an effective theory in order to test the viability of new models. In section 2, I will focus on the former, laying out a systematic method of deriving a low energy effective theory from a known Lagrangian. In section 3 I will turn to an example of using EFT methods to new physics, in a discussion on the use of precision electroweak (PEW) measurements in constraining physics beyond the Standard Model.

# 2 Top-Down Construction of an EFT

## 2.1 A brief outline of the process

In order to construct an effective field theory from a known high energy theory we adopt roughly the following process. First, we identify the relevant light fields (and their symmetries) for the physical domain we want to study. Our aim is to exclude fields heavier than these from direct consideration. To do so we choose a cut-off energy level, and **integrate out** field modes with momenta above this level. In practice the exclusion of these fields involves writing down a general Lagrangian for the light fields, as a sum of all allowed operators, with as yet unknown coefficients.

In general we will find that the higher the order of the operator, the smaller its contribution to low energy physics. This allows us to truncate the sum in order to attain results of a given accuracy. Since our aim is to match experimental results, which are always of finite accuracy, this is sufficient to do physics. This process of determining which operators we need to keep and which we can discard is called **power counting**. It involves examining how the relative contributions of various local operators changes over the energy range of the EFT.

Having accomplished this truncation we need to determine the values of the coefficients. We decide on an observable to calculate, and perform the calculation in both the full and effective theory. In the full theory, the calculation will involve an expansion in powers of the heavy scale. We then **match** the coefficients in the effective theory equal to those given by a specific order in the heavy scale expansion. In this way we incorporate the heavy physics into the couplings of the effective theory in just such a manner as to reproduce the full theory observables to a given accuracy.

The effective theory we have now obtained is a new theory, with new interactions that can be treated perturbatively[2] using Feynman graphs. Note, however, that renormalisability is not a criterion which constrains which operators are "allowed" in our effective Lagrangian so there is no guarantee it is renormalisable. One important conceptual gain from the EFT approach is that we need not be worried by non-renormalisable terms: given that we're only interested in a fixed level of precision, we can calculate in a non-renormalisable theory. Indeed, such terms will capture some important details of the underlying high energy dynamics.

[1] I will make explicit note of where the method outlined here depends on the couplings being weak.

[2] It is possible to construct non-perturbative EFTs, but I shall not consider them.

The new graphs we obtain are typically easier to compute than those of the full theory because they are local: we have exchanged graphs with heavy particle propagators for graphs with local vertices. They are usually *more* divergent than those of the full theory, but still easier to handle as handling the finite parts of graphs is the most intensive portion of the calculation.

If we want to perform loop calculations in our EFT we will have to regulate and renormalise our integrals as in ordinary QFT. Matching takes place order-by-order in a loop expansion, with the effective coefficients from the previous order included in the next. There is some debate about how to renormalise in EFTs, which comes down to two different conceptions of how to do effective field theory. This will be discussed in section 2.5.

## 2.2 Integrating out

In this section I will describe the formalism of "integrating out" heavy modes from a theory[3].

In order to make any headway, we first need to ask what we mean by "heavy". Effective theories require the existence of a scale gap. Suppose our system has a characteristic scale $E_0$ and that we are interested in physics below this level. For example, in particle physics, we might not be interested in very heavy particles like the top quark. We choose a cut-off $\Lambda$ which is smaller than $E_0$. The heavy particles cannot be produced on-shell below $\Lambda$, and as this is as high as we will go when calculating, we will not consider them directly.

Having set the scale $\Lambda$ we proceed to divide the field content $\phi$ of our theory into heavy and light parts. Schematically, we split $\phi = \phi_H + \phi_L$. Typically, the Lagrangians we work with have light fields $\phi$ and heavy fields $\Phi$, and so a natural first step is to write

$$\mathcal{L}[\phi, \Phi] = \mathcal{L}_l[\phi] + \mathcal{L}_h[\Phi] + \mathcal{L}_{lh}[\phi, \Phi], \tag{1}$$

where $\mathcal{L}_l$ ($\mathcal{L}_h$) describes the part of $\mathcal{L}$ involving *only* the light (heavy) fields, and $\mathcal{L}_{lh}$ includes all interactions involving both fields. This does not imply, however, that it is enough to naïvely identify heavy and light fields in the Lagrangian.

There are now two descriptions of how we derive an effective action. In the first we literally perform a path integral over all field modes heavier than the cut-off. In the second (which I will use) we write down an effective Lagrangian which applies below the cut-off, which does not include the heavy fields. I will briefly describe the formal integral and then demonstrate the construction of an effective Lagrangian.

### 2.2.1 Path integral formulation

The Wilson effective action[4] is defined as

$$e^{i\, S_W[\phi_L]} = \int d[\phi_H] e^{i\, S[\phi_L,\, \phi_H]} \tag{2}$$

Note that the field splitting described above requires some careful work. Given $\mathcal{L}[\phi, \Phi]$ with just two fields, and $m_\phi \ll M_\Phi$, it might be tempting to regard the task of generating

[3]This section draws from Polchinski [1] and three of Cliff Burgess' lectures [2, 3, 4].

[4]Named for Ken Wilson whose 1974 paper began the modern study of EFTs.

the Wilson effective action as that of choosing $m < \Lambda < M$ and performing the path integral over $\Phi$:

$$\int d[\Phi]\, e^{i\,S[\phi,\Phi]}.$$

However, we also need to take into account high frequency modes of $\phi$. To see how this works, consider a mode expansion of the light field,

$$\phi(x) = \sum_\ell c_\ell u_\ell(x), \qquad \int d^4x\, u_\ell(x) u_{\ell'}(x) = \delta_{\ell\ell'}, \tag{3}$$

where the basis functions $u_\ell$ are eigenfunctions of the differential operator $\Delta$ that describes the kinetic term in the action:

$$\Delta u_\ell = \omega_\ell u_\ell. \tag{4}$$

These eigenvalues $\omega_\ell$ give precise meaning to the idea of the frequency of the field. So if we write the full path integral as

$$\int d[\phi] d[\Phi]\, e^{i\,S[\phi,\Phi]},$$

then the light field measure is

$$d[\phi] = \prod_\ell dc_\ell, \tag{5}$$

and we can use equation (4) to define high energy modes as those with $\omega_\ell > \Lambda$ and low energy as $\omega_\ell < \Lambda$. We split the measure

$$d[\phi] = \left( \prod_{\substack{\ell \\ \omega_\ell < \Lambda}} dc_\ell \right) \left( \prod_{\substack{\ell \\ \omega_\ell > \Lambda}} dc_\ell \right) \equiv d[\phi]_{\text{l.e.}} d[\phi]_{\text{h.e.}}. \tag{6}$$

The correct definition of the Wilson effective action is then

$$e^{i\,S_W[\phi]} \equiv \int_\Lambda d[\phi]_{\text{h.e.}} d[\Phi]\, e^{i\,S[\phi,\Phi]}. \tag{7}$$

Note that this integral is difficult to perform, and is often done in a perturbative expansion using Feynman diagrams [5] or using a saddle-point approximation [6].

### 2.2.2 Effective Lagrangian formulation

One can derive an effective action without performing a path integral. Instead we write down the most general thing which could be the desired effective Lagrangian, which is to say that we write an infinite sum of all operators involving the light fields which are allowed by the symmetries of the full theory. As $S_W$ is an integral over high energy modes, the uncertainty principle guarantees that the remaining interactions must be *local*. Energies available in the effective theory are low, so high energy effects are produced by local violations of energy conservation. These are allowed only so long as they occur over very small times. The same applies to momentum/distance.

The effective Lagrangian can thus be expanded as

$$\mathcal{L}_{\text{eff}}[\phi] = \mathcal{L}_l[\phi] + \sum_i g_i \mathcal{O}_i[\phi]. \tag{8}$$

with the $\mathcal{O}_i$ local. To see what these operators look like, consider an example. Take the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{M^2}{2}\Phi^2 - \frac{m^2}{2}\phi^2 - V(\phi,\Phi), \tag{9}$$

and choose $m < \Lambda < M$. Our effective Lagrangian will include only $\phi$ and be valid only up to $\Lambda$. We can organise our expansion of $\mathcal{L}_{\text{eff}}$ in powers of derivative terms[5]. The original Lagrangian was invariant under $\phi \to -\phi$ so we will preserve that symmetry, and thus we have:

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + A(\phi) + B(\phi)(\partial_\mu\phi\partial_\mu\phi) \\
&\qquad\qquad + C(\phi)(\partial_\mu\phi\partial_\mu\phi)^2 + \text{other 4 derivative terms } + \ldots \\
A(\phi) &= \frac{1}{4!}a_2\phi^4 + \frac{1}{6!}a_3\phi^6 \ldots \\
B(\phi) &= \frac{1}{2}b_1\phi^2 + \frac{1}{4!}b_2\phi^4 + \ldots \\
&\;\;\vdots
\end{aligned} \tag{10}$$

Note that $A(\phi)$ starts at $a_2$ as it would be pointless to include a constant, and the $a_1$ term has been chosen to reproduce $\mathcal{L}_l$. The same goes for the missing $b_0$ term in $B(\phi)$ which has been fixed to give the kinetic term for $\phi$. Now two questions confront us:

1. In order to achieve fixed order accuracy in $1/M$, which of the above terms do we need? Equivalently, which of the $a_i, b_i, \ldots$ are non-zero?

   **Power counting** answers this question.

2. What are the values of the non-zero coefficients?

   **Matching** answers this question.

Note that there is in general no restriction on how many times we can do this. In a rich theory like the SM we could regard each particle mass as a potential boundary between effective theories. When doing physics we would determine the energy scale of the problem, locate the nearest cut-off and integrate out all particles above that level. As we decrease the energy scale we switch to theories which exclude successively lighter particles. One might imagine a stylised history of particle physics in which new generations of accelerators regularly unlocked new energy regimes, and new physical theories were developed to deal with the new heavy particles discovered. Each new model would need to incorporate (and explain) the results of the previous, lower energy theory. Our tower of EFTs turns this on its head.

## 2.3 Power counting

While we don't know anything about the values of the coefficients at this stage, we can use dimensional analysis to discover things about the relative sizes of the operators. This process of formally estimating the sizes of the operators in $\mathcal{L}_{\text{eff}}$ is called power counting. In simple EFTs, dimensional analysis is a sufficient tool to accomplish this, while for more general purposes[6] we develop an argument based on scaling.

[5] This is equivalent to a momentum power expansion.

[6] In complex EFTs power counting is renormalisation scheme-dependent, but we do not consider such complications here.

### 2.3.1 Naïve dimensional analysis

I'll demonstrate the dimensional analysis argument in $D$ dimensions[7] and then work through a scaling example for $D = 4$. We work in natural units ($\hbar = c = 1$) so that [mass]=[energy]=[length]$^{-1}$. All dimensions are expressed in units of [mass] so that we simply refer to the exponent as the "dimension". Now $\int d^D x \mathcal{L}$ must be dimensionless and $[d^D x] = -D$, so $\mathcal{L}$ has mass dimension $D$. Each term in $\mathcal{L}$ must therefore also have dimension $D$. We calculate the dimensions of the fields from the kinetic term in the Lagrangian[8]. So for a scalar field we have $\partial_\mu \phi \partial^\mu \phi$, and as $[\partial_\mu] = 1$, $[\phi] = D/2 - 1$. Fermionic kinetic terms look like $i\bar{\psi}\partial\!\!\!/\psi$, so we have $[\psi] = (D-1)/2$.

Once we have the dimension of the fields, we can easily arrive at the dimensions of the operators. By the same logic as above, if the dimension of operator $\mathcal{O}_i$ is $\delta_i$ then its coupling $g_i$ has dimension $D - \delta_i$. For convenience we define dimensionless couplings

$$\lambda_i \equiv \frac{g_i}{\Lambda^{D-\delta_i}} \tag{11}$$

where $\Lambda$ is the fixed cut-off, the largest scale in the effective theory.

So, given the dimension of an operator we can estimate its magnitude. If all processes are taking place at energy scale $E < \Lambda$ then this sets the size of terms in the action:

$$\int d^D x \, g_i \mathcal{O}_i \sim \lambda_i \left(\frac{E}{\Lambda}\right)^{\delta_i - D}. \tag{12}$$

This is the core of power counting by dimensional analysis. We see that as $E$ shrinks we can estimate the size of different operators, depending on their dimension.

| dimension | size as $E \to 0$ | name |
|---|---|---|
| $\delta_i < D$ | grows | relevant |
| $\delta_i = D$ | constant | marginal |
| $\delta_i > D$ | shrinks | irrelevant |

Table 1: Operator classification by dimension.

### 2.3.2 Scaling

It will be simplest to explain the scaling argument behind power counting by using an example. The following is due to Kaplan [7]. Consider the effective field theory described by $\mathcal{L}_{\text{eff}}$ in equation (10), consisting of one scalar field $\phi$ and preserving a $\phi \to -\phi$ symmetry. We begin by expressing $\mathcal{L}_{\text{eff}}$ in a more familiar form (adopting Euclidean signature) and pulling out a $\phi^4$ term for the purposes of later analysis.

$$\mathcal{L}_{\text{eff}} = +\frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \sum_n \left( \frac{a_n}{\Lambda^{2n}}\phi^{4+2n} + \frac{b_n}{\Lambda^{2n}}\phi^{2+2n}(\partial\phi)^2 + \dots \right). \tag{13}$$

The couplings have been rescaled according to the prescription above, and the $2n$ powers in the sum ensure we preserve the symmetry.

[7] Following Polchinski [1].

[8] This determination sets the "engineering dimension" of the fields, and only works for weakly coupled theories.

Suppose we scale the field, by taking $\phi(x) \to \phi_s(x) = \phi(sx)$. This would yield

$$S_W[\phi_s(x)] = \int d^4x \;\frac{1}{2}(\partial_x\phi(sx))^2 + \frac{m^2}{2}\phi(sx)^2 + \frac{\lambda}{4!}\phi(sx)^4 + \sum_n \left(\frac{a_n}{\Lambda^{2n}}\phi^{4+2n}(sx) + \frac{b_n}{\Lambda^{2n}}(\partial_x\phi(sx))^2\phi^{2+2n}(sx) + \dots\right).$$

Now define a new variable $y = sx$. Then $d^4x = s^{-4}d^4y$, $\partial_x = s\partial_y$. Also define $\tilde{\phi}(y) = s^{-1}\phi(y)$. Then we have

$$S_W[\tilde{\phi}(y)] = \int d^4y \;\frac{1}{2}(\partial_y\tilde{\phi}(y))^2 + \frac{m^2s^{-2}}{2}\tilde{\phi}(y)^2 + \frac{\lambda}{4!}\tilde{\phi}(y)^4 + \sum_n \left(\frac{a_n s^{2n}}{\Lambda^{2n}}\tilde{\phi}^{4+2n}(y) + \frac{b_n s^{2n}}{\Lambda^{2n}}(\partial_y\tilde{\phi}(y))^2\tilde{\phi}^{2+2n}(y) + \dots\right). \tag{14}$$

Since $y$ is an integration variable, we can recognise that equation (14) gives the same action as that derived from (13), but with the following scalings:

$$\begin{aligned} \phi &\to s^{-1}\phi, \\ m^2 &\to s^{-2}m^2, \\ \lambda &\to \lambda, \\ a_n &\to s^{2n}a_n, \\ b_n &\to s^{2n}b_n. \end{aligned}$$

Now as we scale to the low energy processes by taking $s \to 0$ we see the behaviour shown in Table 2.

| coefficient | size as $s \to 0$ | name |
|---|---|---|
| $m^2$ | grows | relevant |
| $\lambda$ | constant | marginal |
| $a_n, b_n$ | falls | irrelevant |

Table 2: Operator classification by scaling.

A few comments:

1. We scale $\phi \to s^{-1}\phi$ in order to keep the kinetic term scale invariant. This is because, for a weakly interacting system, this term dominates the size of the fluctuations in the path integral. This argument is only valid for relativistic theories, but this need not concern us here.

2. What we have learnt from the scaling behaviour of the $a_n$ and $b_n$ terms is that higher terms (larger $n$) are more irrelevant. So my introductory remark that higher order operators are suppressed has been vindicated. Therefore we are now justified in believing that we can attain fixed order results by truncating this infinite sum. Note that we have assumed that this theory is weakly coupled ($\lambda, a_n, b_n \ll 1$) and can therefore be treated perturbatively[9].

[9] If the high energy theory is strongly coupled, effective theories work differently. Instead of aiming to calculate the effective theory's couplings in terms of parameters from the high energy theory, we treat the low energy parameters as free, to be fixed by experiment.

3. Note, however, that the term "irrelevant" is somewhat misleading. These terms should not simply be ignored and indeed often contain important information about the underlying high energy dynamics.

### 2.3.3 Naturalness

There is one further constraint on effective theories, beyond symmetries and locality of operators, which had to wait on an explanation of power counting. This is the constraint of "naturalness", which is variously described as anything from an aesthetic constraint to a major problem for a theory. For a theory to be natural, its dimensionless couplings ($\lambda_i$) should be of order 1, while the dimensionful couplings ($g_i$) should be of the order of the heavy scale ($M$). Looking at how I defined the $\lambda_i$ above, it is clear that both can be satisfied. However, there is a problem arising from this, which concerns mass terms. Consider the $\phi^2$ term, which has dimension 2 for $D = 4$. The dimensionless coupling $\lambda$ for this term will be of order 1, which means the dimension-cancelling $\Lambda^2$ will set the scale. But now the Lagrangian contains

$$\frac{1}{2}m^2\phi^2 = \frac{1}{2}(2\lambda\Lambda^2)\phi^2,$$

a mass term for $\phi$ with the mass of the order of $\Lambda$ — the cut-off! If this is so, this field should not be in our effective theory at all.

Therefore, in order for a theory to be natural it cannot contain mass terms. More precisely, natural theories are those where all mass terms are forbidden by symmetries. Note that this does not mean our effective theories cannot deal with physically massive objects. It simply means that masses can only arise from spontaneously broken high-energy symmetries.

The Standard Model is not natural. While we could explain the masses of the $W$ and $Z$ bosons as being due to the broken electroweak symmetry (the fields they originate from are massless prior to SSB), there is no symmetry forbidding a mass term for the Higgs.

## 2.4 Tree-level matching

Matching too is easiest explained using an example. This provides the first opportunity to apply some of the ideas developed so far. The aim of a matching calculation is to fix the values of the effective coefficients in $\mathcal{L}_{\text{eff}}$ so that we reproduce the predictions of the full theory to fixed accuracy. This is one of the two ways in which the high-energy theory determines the content of the effective theory, with the other being the application of symmetry constraints.

Following Burgess [2] I will use the classic complex scalar field Lagrangian with SSB, as it provides us with naturally separated fields — a light (massless) Goldstone boson and a heavy field.

$$\mathcal{L} = -\partial_\mu\phi^*\partial^\mu\phi - \frac{\lambda^2}{4}\left(\phi^*\phi - v^2\right)^2 \tag{15}$$

Written in the form of equation (15) it is clear that we have a symmetry $\phi \to \epsilon^{i\omega}\phi$, for $\partial_\mu\omega = 0$. In order to highlight this we adopt the field redefinition

$$\phi \equiv \chi e^{i\theta} \tag{16}$$

so that (15) becomes

$$\mathcal{L} = -\partial_\mu\chi\partial^\mu\chi - \chi^2\partial_\mu\theta\partial^\mu\theta - \frac{\lambda^2}{4}\left(\chi^2 - v^2\right)^2. \tag{17}$$

The structure of the theory is now clearer. We see that we have two fields; $\theta$, which is massless, and $\chi$ with mass $M = \lambda v$. All we have done is taken into account the shape of the potential (shown in Fig. 1) and adopted a convenient field parametrisation. $\chi$ is a radial coordinate, moving us up and down the slopes of the Mexican hat; $\theta$ is an angular coordinate, so when we're at the degenerate minimum (represented by the trough) we can translate around that circle for no energy cost by changing $\theta$.

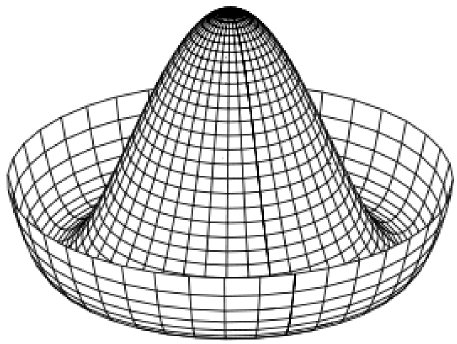
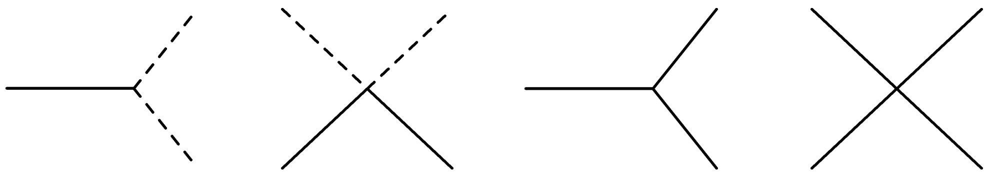


Figure 1: Mexican hat potential described by (15).

For the purposes of normalisation I will redefine the fields $\chi \to v + \frac{1}{\sqrt{2}}\psi$, $\theta \to \frac{1}{\sqrt{2}v}\xi$. Then the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}v}\psi\right)^2\partial_\mu\xi\partial^\mu\xi - \frac{\lambda^2}{4}\left(\sqrt{2}v\psi + \frac{1}{2}\psi^2\right)^2. \tag{18}$$

Fig. 2 shows the vertices this produces.



(a) $\psi\partial_\mu\xi\,\partial^\mu\xi$ (b) $\psi^2\partial_\mu\xi\,\partial^\mu\xi$ (c) $\psi^3$ (d) $\psi^4$

Figure 2: Interactions in the scalar theory described by equation (18).

For the purposes of constructing our effective theory we will need to choose some observable to calculate. Let's use $\xi\xi \to \xi\xi$ scattering, which occurs at tree-level in the full theory through the diagrams shown in Fig. 3. We get $s$, $t$ and $u$ channel processes, all formed from the $\psi\partial_\mu\xi\partial^\mu\xi$ vertex. We will assign momenta to the external lines as follows: $p$ and $q$ to incoming lines, $p'$ and $q'$ to outgoing, and write the mass of the $\psi$ as $M = \lambda v$. Then the amplitude is trivial to write down:

$$\begin{aligned}\mathcal{A}_f &= \frac{(-)^2}{2!}4\cdot 2\cdot\left(\frac{1}{\sqrt{2}v}\right)^2\left[\frac{(p\cdot q)(p'\cdot q')}{(p+q)^2+M^2} + \frac{(p\cdot p')(q\cdot q')}{(p-p')^2+M^2} + \frac{(p\cdot q')(p'\cdot q)}{(p-q)^2+M^2}\right] \\ &= \frac{2}{v^2}\left[\frac{(p\cdot q)(p'\cdot q')}{(p+q)^2+M^2} + \frac{(p\cdot p')(q\cdot q')}{(p-p')^2+M^2} + \frac{(p\cdot q')(p'\cdot q)}{(p-q)^2+M^2}\right]\end{aligned} \tag{19}$$

We can tidy this up with some algebra. As $p+q = p'+q'$ we know that for a massless $\xi$ we have (squaring both sides) $p \cdot q = p' \cdot q'$. Similar equalities emerge from rearranging the above, and we end up with:

$$\mathcal{A}_f = \frac{2}{v^2}\left[\frac{(p\cdot q)^2}{(p+q)^2+M^2} + \frac{(p\cdot p')^2}{(p-p')^2+M^2} + \frac{(p\cdot q')^2}{(p-q)^2+M^2}\right] \tag{20}$$

The effective theory we will construct will be accurate to a given order in $1/M$, so we now expand this amplitude as a series in $1/M$. This is simply done by writing

$$\frac{1}{X^2+M^2} = \frac{1}{M^2}\left(1 - \frac{X^2}{M^2} + \frac{X^4}{M^4} + \dots\right)$$

If we wish to work to $\mathcal{O}(1/M^2)$ then we simply have:

$$\mathcal{A}_f = \frac{2}{v^2M^2}\left[(p\cdot q)^2 + (p\cdot p')^2 + (p\cdot q')^2\right] + \mathcal{O}\left(\frac{1}{M^2}\right) \tag{21}$$

We now need to construct the effective Lagrangian for $\xi$, and calculate the same amplitude using $\mathcal{L}_{\text{eff}}$. We will then compare the two, and match the coefficient from the effective theory to that implied by the full result.

The full theory provides us with a symmetry constraint. Phrased in terms of the old field $\phi$ it was an overall phase multiplication. In terms of $\xi$ this is a translation by some $\omega$ such that $\partial_\mu\omega = 0$. Therefore we want operators built from $\partial_\mu\xi$ but not simply $\xi$. The lowest order Lagrangian containing $\xi\xi \to \xi\xi$ scattering is therefore

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2}\partial_\mu\xi\partial^\mu\xi - a\big(\partial_\mu\xi\partial^\mu\xi\big)^2 + \dots \tag{22}$$

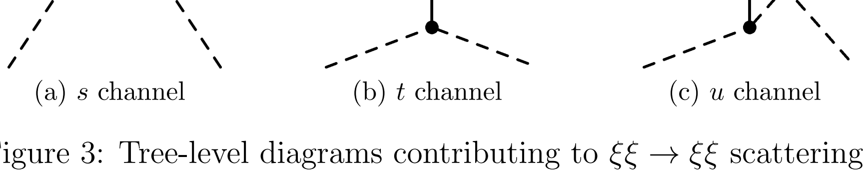
As expected for an effective theory, we have a local interaction which just is tree-level $\xi\xi \to \xi\xi$ scattering in this theory. Noting the derivatives on the $\xi$ fields, and that they're contracted, we get the following matrix element:

$$\begin{aligned}\mathcal{A}_{\text{eff}} &= -a\cdot 4\cdot 2\big[(p\cdot p')(q\cdot q') + (p\cdot p')(q\cdot q') + (p\cdot q')(p'\cdot q)\big] \\ &= -8a\big[(p\cdot q)^2 + (p\cdot p')^2 + (p\cdot q')^2\big]\end{aligned} \tag{23}$$

So comparing equations (21) and (23) we can see that if we set

$$a = \frac{1}{4v^2M^2} \tag{24}$$

then we reproduce the result obtained in the full theory to finite accuracy (here, order $1/M^2$). This is a simple example of a "matching" calculation. By matching the coefficient in the effective theory to that produced (approximately) by the full theory we embed information about the heavy field $\psi$, which is *not* itself part of this theory, into our results. In this example, there is not much interesting in our effective theory, but in general we could now proceed on to calculate other observables.



(a) $s$ channel (b) $t$ channel (c) $u$ channel

Figure 3: Tree-level diagrams contributing to $\xi\xi \to \xi\xi$ scattering.

## 2.5 Renormalisation

We often want to go beyond mere tree-level calculations. To do so, we'll need to decide on a renormalisation scheme to handle loop calculations. This decision, usually one of convenience, is more loaded in EFT.

### 2.5.1 Renormalisability and EFT

One of the major historical impacts of EFT has been a revolution in how we regard renormalisation as a feature of quantum field theories[10]. The notion of effective field theory emerged from the study of the renormalisation group and many of the key results underpinning its development are results in renormalisation theory. In particular, Appelquist and Carazzone proved the following theorem in 1975:

> Given a renormalisable theory in which some fields have masses much larger than the others, a renormalisation prescription can be found such that the heavy particles decouple from the low energy physics, except for producing renormalisation effects and corrections that are suppressed by a power of the experimental momentum over a heavy mass.

The Appelquist-Carazzone decoupling theorem implies that the low energy physics is describable by an effective theory which incorporates only the light fields, and is thus the core of the EFT approach.

While renormalisation had an unenthusiastic reception, it soon became accepted as a useful calculation tool. Indeed, it was the only way of successfully calculating, and thus renormalisability was taken as a *criterion* for building good theories. In the effective theory understanding, this is approach is misguided. Non-renormalisable terms are acceptable so long as we recognise that our theories are built from a position of ignorance and should therefore be regarded as valid only up to a cut-off. Indeed, non-renormalisable terms tell us where we need the cut-off. When they stop describing the what we see, it is because some new physics has emerged which requires a new (effective) theory.

As an example, consider Fermi's theory of the weak interaction. This is a non-renormalisable low energy approximation to (renormalisable) electroweak theory which postulates a four-fermion vertex. It functions up to a cut-off (about 300GeV) set by the heavy particles it excludes. If we conduct experiments near this energy level we should expect a breakdown of Fermi theory (which is quite accurate below this level). Fermi theory must now be replaced by a new theory which incorporates the heavier fields.

This new view of model building raises questions about those theories which were built using renormalisability as a criterion. Quantum Electrodynamics was written down as the most general renormalisable Lagrangian which is $U(1)$ invariant. How do we explain its remarkable success if (excluded) non-renormalisable terms are acceptable? Consider electron-positron scattering. The answer is that the lowest-dimensional non-renormalisable contribution is due to $Z$-exchange. But $m_Z$ is very large, and the leading non-renormalisable contributions are suppressed by $1/m_Z^2$. It is this suppression and not renormalisability which makes QED such a good theory of electron-positron scattering.

[10]This section is based on Polchinski [1] and Cao & Schweber [8]

### 2.5.2 Renormalisation schemes and EFTs

A renormalisation scheme consists of a method for making infinite integrals finite (a regulator) and a method for dealing with the infinite part (a subtraction scheme). These can broadly be classified in terms of the mass-dependence of the subtraction scheme. Physical quantities cannot depend on our choice of renormalisation scheme, so this is usually a choice of convenience of preference. In the context of effective field theory, however, Bain [6] and Georgi [9, 10] argue that the choice implies a conception of how EFTs function. The choice is broadly between mass-dependent schemes (which go with a 'Wilsonian' formulation of EFT) and mass-independent schemes (which go with Georgi's 'continuum EFT' formulation).

The presence of the effective theory cut-off $\Lambda$ makes a momentum cut-off regularisation seem appropriate. This directly implements the idea that in EFT we calculate only below $\Lambda$. Furthermore, as Georgi [10] notes, the Appelquist-Carazzone theorem requires a mass-dependent scheme to operate: in a mass-independent scheme "the renormalisation scale dependence [in, e.g., SU(5)] of the Feynman graphs are the same whether there is a gluon or an $X$ particle inside." So the decoupling which in some sense *justifies* the EFT is only formally guaranteed under such a scheme.

However, there are a number of disadvantages to mass-dependent renormalisation. Firstly, such schemes violate Poincaré and gauge invariance. To see this, note that we can Fourier transform our regulated momentum-space integral, and arrive at a coordinate space integral over a finite region. It is this volume restriction which violates Poincaré invariance, and its absence which lends Georgi's formulation the name "continuum" EFT.

Secondly, mass-dependent schemes make loop calculations in EFTs exceedingly difficult, as the mass-dependence leads to large dimensionful factors appearing in the numerators of higher-order terms, disrupting the suppression by inverse powers of the cut-off. The formal and principled reasons for adopting such a scheme are, to Georgi at least, outweighed by how difficult they make the business of doing physics.

Mass-independent schemes, such as dimensional regularisation and modified minimal subtraction, correct for these latter flaws. They preserve the symmetries of the higher theory, and lead to logarithmic (but not power law) dependence on the heavy scale. Thus, we can count powers of $1/\Lambda$ as described above and are guaranteed suppression of irrelevant terms.

How do we deal with the lack of guaranteed decoupling? The answer to this is what motivates Bain to describe the choice of renormalisation scheme as a conceptual disagreement about how EFT works. In the Wilsonian formulation decoupling underlies the formation of an EFT. For Georgi, it is "put in by hand in matching." The two formulations also lead to the major calculation difference described in section 2.2; in Wilsonian EFT the effective action is defined by a path integral, while in continuum EFT we simply write down $\mathcal{L}_{\text{eff}}$.

### 2.5.3 Higher-order matching

In this section I will sketch how matching works at higher orders in a loop expansion, working with in a mass-independent scheme[11]. At one-loop level, we will need to regularise both the full and effective theory using dimensional regularisation. In the full theory, we renormalise using $\overline{\text{MS}}$. In the effective theory we renormalise by demanding that the

[11]This section is based on Pich [11] and Kaplan [12].

renormalised coupling coefficients match those in the full theory. Matching proceeds in this way, order by order. The previous order effective coupling constant is that which is modified by the renormalisation-matching calculation.

Let's start by taking the theory described by equation (9) and choosing $V(\phi, \Phi) = \lambda/2\phi^2\Phi$.

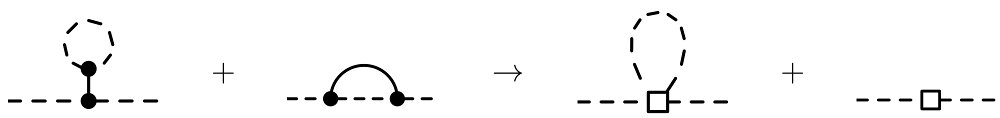
$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{M^2}{2}\Phi^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{2}\phi^2\Phi \tag{25}$$

In the full theory $\phi\phi \to \phi\phi$ scattering occurs via $\Phi$ exchange. In our effective theory, this is replaced by the $\phi^4$ term. The effective Lagrangian is that of (13). I'll write the first few terms below, using general coefficients:

$$\mathcal{L}_{\text{eff}} = \frac{a}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{b}{2}\phi^2 + \frac{c}{4!}\phi^4 + \ldots \tag{26}$$

At tree-level, $a = 1, b = m^2$ and the matching calculation for $c$ is essentially similar to that done in section 2.4 and results in $c = 3\lambda/M^2$.

At 1-loop, the matching condition is that we replace the diagrams on the lhs of equation (27) with those on the rhs. The first diagram on the right represents zeroth loop order and the second represents first loop order.


(27)

Without doing the calculation, we can sketch the outline of the result using some basic facts about renormalisation calculations and power counting. We know that each loop will yield a factor of $1/16\pi$. Corrections to the tree level coefficients much match their dimensions. From equation (25) we can see $\lambda$ has dimension 1, and therefore $a$ and $c$ are dimensionless while $b$ has dimension 1. Loop corrections must therefore also have these dimensions. Using $\lambda$ to fix dimensions, we therefore have

$$\begin{aligned} a &= 1 + a_1\frac{\lambda^2}{16\pi^2 M^2} + \ldots \\ b &= m^2 + b_1\frac{\lambda^2}{16\pi^2} + \ldots \\ c &= \frac{3\lambda}{M^2} + c_1\frac{\lambda^2}{16\pi^2 M^2} + \ldots \end{aligned}$$

$a_1, b_1$ and $c_1$ are dimensionless numbers which must be matched by comparing the amplitudes for $\phi\phi \to \phi\phi$ scattering at 1-loop in the full and effective theory. In the full theory the calculation must be done as usual, using dimensional regularisation and $\overline{\text{MS}}$. In the effective theory, we regulate and then renormalisation is accomplished *by* matching at the scale $\mu = M$. We then renormalise the fields to canonically normalise the kinetic energy term.

# 3 Bottom-Up EFTs and the PEW Measurements

## 3.1 Introduction

We now turn to the application of effective field theories where the UV theory is not known[12]. The main application is in extending our physical theories beyond the Standard Model (SM). The basic idea is simple. We know that below the electroweak scale the SM is a good theory, showing excellent agreement with experiment. We will use it as a base to construct an effective theory that includes the SM Lagrangian and an infinite sum of local operators.

Suppose $\mathcal{L}_{\text{BSM}}$ represents the unknown UV theory, consisting of the SM field content $\phi_{\text{SM}}$ and new fields $\varphi$[13]. Then if we suppose that the new fields are heavier than $\phi_{\text{SM}}$ (which is reasonable as they represent undetected particles) we know we can write an effective theory in the manner of equation (8):

$$\mathcal{L}_{\text{BSM}}[\phi_{\text{SM}}, \varphi] \to \mathcal{L}_{\text{eff}}[\phi_{\text{SM}}] = \mathcal{L}_{\text{SM}} + \sum_i g_i \mathcal{O}_i(\phi_{\text{SM}}). \tag{28}$$

Note a few important features equation (28). First, it contains the Standard Model as $\mathcal{L}_l$; this is required as we want to reproduce our current results by *extension* of the SM. Second, as we've come to expect, the effective Lagrangian does not dependent on the new field content $\varphi$ of $\mathcal{L}_{\text{BSM}}$ as these fields have been integrated out. This is important here because it means the methods we will discuss here are independent of any particular model of physics beyond the Standard Model (BSM). Our aim is to discuss a general enough framework that one could simple "plug in" a favoured model.

In doing this we have to make a number of assumptions:

1. The new physics decouples, in the manner suggested by the Appelquist-Carazzone theorem. i.e. The heavy fields decouple in the limit $\Lambda \to \infty$ where $\Lambda$ is a cut-off initially set at the characteristic scale of the new physics.

2. Since the physics of electroweak symmetry breaking is not experimentally confirmed we have to make an assumption about the Higgs sector. For the purposes of this paper I will assume that EW symmetry is broken by a Higgs doublet. What this means is outlined below in section 3.2.1.

The core work in applying EFT techniques to analysing potential models of new physics is in seeing how those models would contribute to various local operators.

In the rest of this section I will look at the use of the precision electroweak measurements to constrain a class of effective operators called "oblique" operators. As this requires some facility with the physics of the SM, I begin with some background. I then briefly discuss the PEW measurements, motivating their use. I then turn in section 3.3 to discussing how dimension 6 oblique operators constrain additions to the SM. Finally, in section 3.4 I give a high level discussion of one model of new physics and how can be constrained using these methods.

[12] This introductory section is based on the introduction in de Blas Mateo [13] and Skiba [5].

[13] Note that there is no guarantee that the new theory will be formulated in terms of the familiar SM physical fields, or the $SU(2) \times U(1)$ gauge fields. There could be higher symmetries which break spontaneously to yield the SM fields. But up to such field redefinitions the above statement is a reasonable way of thinking about how a UV theory (of which the SM is a low energy effective theory) would be constituted.

## 3.2 Background

In this section I explain various pieces of the physics of the Standard Model which are required for a meaningful discussion of using EFTs to constrain new physics.

### 3.2.1 Electroweak symmetry breaking

In order to explain tests of physics beyond the Standard Model[14], it will be useful to review some of of the physics of the electroweak sector, in particular electroweak symmetry breaking (EWSB). This portion of the SM is all that still requires experimental confirmation. The theoretical addition which accomplishes EWSB in the SM is the "Higgs mechanism", which introduces a doublet of scalar fields to the SM Lagrangian:

$$H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix},$$

with Lagrangian

$$\mathcal{L}_{\text{Higgs}} = (\mathcal{D}_\mu H)^\dagger (\mathcal{D}^\mu H) - \mu^2 H^\dagger H + \lambda (H^\dagger H)^2.$$

This field has a non-zero vacuum expectation value (vev). $H$ is expanded around the vev, which yields

$$H = \mathcal{U} \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}.$$

The $\mathcal{U}$ is a unitary term which turns out to be unimportant. The upshot of this is that the expansion about the Higgs vev brings about spontaneous symmetry breaking. The covariant derivative $\mathcal{D}_\mu$ contains interactions with the electroweak gauge fields:

$$\mathcal{D}_\mu = \partial_\mu + \frac{i}{2} g_1 B_\mu + \frac{i}{2} g_2 \tau^a W_\mu^a \tag{29}$$

The result is that the Higgs Lagrangian can be expanded out as

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} = \frac{1}{2}\partial_\mu h \partial^\mu h &+ \frac{1}{8}(g_1^2 + g_2^2)(B_\mu - W_{3\mu})(B^\mu - W_3^\mu)(v+h)^2 \\ &+ \frac{g_2^2}{8}(W_{1\mu} - iW_{2\mu})(W_1^\mu - iW_2^\mu)(v+h)^2 \\ &+ \text{terms involving just } h \end{aligned}$$

From the second and third terms we can identify physical fields made from combinations of the gauge fields $B_\mu$ and $W_{a\mu}$. This is done by expanding the $(v+h)^2$ and looking for terms with the form of mass terms. This leads to a set of field redefinitions which take us from the gauge fields to the physical fields of the photon, $W^\pm$ and $Z$ boson. For the photon and $Z$ boson we can represent the transformation from the gauge fields to the physical fields as a rotation through an angle $\theta_W$ called the Weinberg weak mixing angle:

$$\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix} \tag{30}$$

This angle is defined by

$$\sin^2\theta_W = \frac{g_1^2}{g_1^2 + g_2^2} \tag{31}$$

[14]This section is based on my notes from Dr Matthew Wingate's course on the Standard Model as well as Dr Hugh Osborn's set of notes [14].

Writing $s_W = \sin\theta_W$ and $c_W = \cos\theta_W$, the field redefinitions are

$$\begin{aligned} A_\mu &\equiv c_W W_{3\mu} + s_W B_\mu, \\ Z_\mu &\equiv c_W W_{3\mu} - s_W B_\mu, \\ W_\mu^- &\equiv \frac{1}{\sqrt{2}}(W_{1\mu} + iW_{2\mu}), \\ W_\mu^+ &\equiv \frac{1}{\sqrt{2}}(W_{1\mu} - iW_{2\mu}). \end{aligned}$$

This leads to the following relation between the masses of the $Z$ and $W$ bosons

$$\frac{m_W}{m_Z} = \cos\theta_W, \tag{32}$$

which can be written in terms of the Higgs vev and gauge couplings as

$$m_Z^2 = \frac{v^2}{4}(g_1^2 + g_2^2), \tag{33}$$

$$m_W^2 = \frac{g_2^2 v^2}{4}. \tag{34}$$

#### 3.2.2 Custodial symmetry

We can convert the above relation between $W$ and $Z$ into a single parameter $\rho$ defined as

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W}. \tag{35}$$

At tree-level in the SM we know that $\rho = 1$. It is possible that there are higher order radiative corrections to this, which would arise from processes which shift the $Z$ mass without changing the $W$ mass (or vice versa). It is possible to theoretically disallow such terms by positing a larger global symmetry group than is currently part of the SM.

The EW sector has the symmetry breaking structure

$$SU(2)_L \times U(1)_Y \to U(1)_{\mathrm{QED}}.$$

Under this new proposal, we introduce a symmetry which groups right-handed fermions into $SU(2)$ doublets. This $SU(2)_R$ symmetry is global. We gauge only the $U(1)$ portion of it and are left, after symmetry breaking, with $SU(2)_{\mathrm{cust}} \times U(1)_{\mathrm{QED}}$. This additional symmetry explicitly forbids terms which change $\rho$ from unity.

This has led to a habit of referring to any new physics which results in $\rho \neq 1$ as "breaking custodial symmetry". Below, this should be taken to mean only that these processes shift the mass of either the $Z$ or $W$ bosons, and nothing more.

#### 3.2.3 Vacuum polarisation

The particular type of new physics discussed below affects the self-energies of the $W$ and $Z$ bosons[15]. In QFT, propagators receive loop corrections due to the production of fermion–anti-fermion loops, as shown in Fig. 4. This leads to a phenomenon known as vacuum polarisation. As the fermion and anti-fermion carry opposite charge, they form a dipole as

[15] Formulae in this section are taken from Ryder [15].

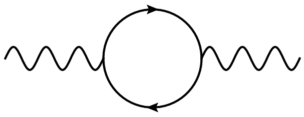


Figure 4: A standard vacuum polarisation diagram.

they run round the loop. This dipole produces a screening effect on the background field, weakening it and making the presence of the virtual particles in the loop experimentally detectable.

If we take Fig. 4 to be a photon self-energy diagram, and give the incoming photon momentum $k$, the upper leg $p$ and the lower leg $p-k$ we get the following for its self-energy, also called its vacuum polarisation function:

$$i\Pi_{\mu\nu}(k) = -(ie)^2 \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left(\gamma_\mu \frac{i}{\not{p}-m} \gamma_\nu \frac{i}{\not{p}-\not{k}-m}\right) \tag{36}$$

This corrects the propagator, so that at one loop level

$$i\Delta_{\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2} + \left(\frac{-ig_{\mu\alpha}}{k^2}\right) i\Pi_{\alpha\beta}(k) \left(\frac{-ig_{\beta\nu}}{k^2}\right). \tag{37}$$

### 3.2.4 Precision electroweak measurements

The most stringent tests of the Standard Model, as well as the best constraints on extensions to it, come from a series of experiments referred to as the "precision electroweak measurements". These measurements were performed at the Large Electron Positron (LEP) collider at CERN and the Stanford Linear Collider (SLC) at SLAC. These experiments focused on the $e^-e^+ \to f\bar{f}$ decay (one channel of which is $e^+e^- \to Z \to f\bar{f}$). These are referred to as "$Z$-pole" measurements, for the spike in the $e^+e^-$ cross-section about the $Z$ mass (see Fig. 5[16]), and measured (amongst other things) the mass and decay width of the $Z$ boson.

The PEW data allow for a large number of tests of the predictions of the SM. To take one small example, the $Z$-pole data can be used to derive the number of generations of neutrinos, which is found to be $N_\nu = 2.994 \pm 0.011$, very close to the SM's three. Critically, the PEW data allows us to fix two of the three free parameters in low energy electroweak physics: the fine-structure constant, Fermi's constant, and the electroweak mixing angle. Table 3 shows the $W$ and $Z$ masses[17].

Given the effective theory of equation (28) we need to determine which operators are relevant and what the values of their coefficients are (i.e. power count and then match coefficients). Recall that we plan to fix these coefficients by comparison with experiment. Thus, what determines relevance is the precision of the experimental data with which we are comparing. Equation (12) states that an operator of dimension $\delta$ will contribute $\sim (E/\Lambda)^{\delta-4} < 1$, for $E$ the energy characteristic of some process in our effective theory. Let $\sigma$ be the finite experimental precision of our data. Then by the above argument we

[16] Figure from Ref. [16].

[17] All Particle Data Group figures are taken from the latest review [17].

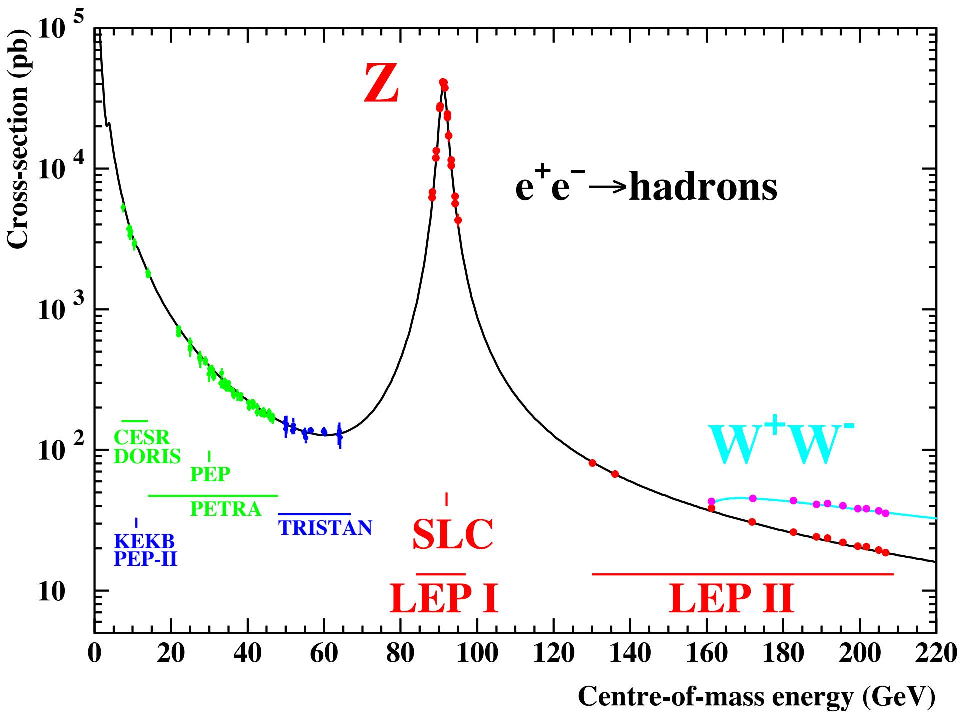



Figure 5: The hadronic cross-section as a function of centre-of-mass energy. Note the $Z$ pole.

| Quantity | Exp. value | SM value |
|---|---|---|
| $m_Z$ [GeV] | $91.1876 \pm 0.0021$ | $91.1874 \pm 0.0021$ |
| $m_W$ [GeV] | $80.420 \pm 0.031$ | $80.384 \pm 0.014$ |
| | $80.376 \pm 0.033$ | |

Table 3: PDG figures for $Z$, $W$ mass in experiment and theory. The first $m_W$ figure is from the TeVatron, the second is from LEP2. Note that neither of the latter two are PEW measurements.

need only consider operators of dimension $\delta \leq N + 4$ where $N$ is fixed by

$$\sigma \sim \left(\frac{E}{\Lambda}\right)^N . \tag{38}$$

The effects of higher order operators are beyond the precision of the experimental data we are comparing to, and are therefore not worth computing. The methods described here obviously work only if the experimental accuracy is greater that the first set of operators not in the low energy theory (i.e. dimension 5 operators).

We can use the above argument to set rough bounds on our effective Lagrangian easily. For example, if we assume that new physics occurs at about the TeV scale, and want to work with $Z$-pole measurements (explained later) then taking $m_Z$ to be roughly 91GeV we get $E/\Lambda = m_Z/TeV \approx 0.09$. So for $N = 2$ we require experimental precision at the $\sigma = 0.83\%$ level, while for $N \geq 3$ we require precision $\leq 0.08\%$. Given that Skiba loosely says that the precision electroweak measurements are accurate to "about 1%" it is clear why EFT analyses of the PEW data consider mostly dimension 6 operators. That said, specialising to the most accurate measurements in the data set can allow for precision an order of magnitude greater, allowing us to stretch our analysis to higher-order operators. By contrast, LEP2 measurements of the $W$ mass are an order of magnitude less precise.

## 3.3 Effective operators

### 3.3.1 The Peskin-Takeuchi parameters

In a 1991 paper Peskin and Takeuchi derived a set of three parameters, labelled $S$, $T$ and $U$ which capture the effect of new physics to the electroweak sector. Roughly speaking $S$ parameterises the 'size' of the new sector and $T$ captures the total weak-isospin breaking it introduces. $S$ and $T$ parameterise the effects of dimension six operators, which will be considered here, while $U$ parameterises effects due to dimension eight operators, and will therefore not be discussed.

We can make the above meaning of $T$ more specific. $T$ breaks custodial symmetry directly:

$$\rho = 1 + \delta\rho_{\rm SM} + \alpha T,$$

where $\delta\rho_{\rm SM}$ term is to account for any high loop corrections which may be calculated. The PDG currently lists the following as the measured value of $\rho$:

$$\rho_{\rm PEW} = 1.0008^{+0.0017}_{-0.0007}.$$

In deriving $S, T$ and $U$ a few assumptions are made about the new physics sector:

1. The electroweak gauge group is unaffected by the new physics (so, in particular, it does not include new EW bosons).

2. The new fields do not couple to SM fermions. This condition defines an "oblique" operator.

3. The energy scale at which the new physics appears is large compared to the electroweak scale.

A clear definition of these parameters (taken from Hewett [18]) in terms of vacuum polarisation functions of the electroweak bosons is

$$\alpha S = 4s_W^2 c_W^2 \big(\Pi'_{ZZ}(0) - \frac{c_W^2 - s_W^2}{s_W c_W}\Pi'_{Z\gamma}(0) - \Pi'_{\gamma\gamma}(0)\big), \tag{39}$$

$$\alpha T = \big(\frac{1}{m_W^2}\Pi_{WW}(0) - \frac{1}{m_Z^2}\Pi_{ZZ}(0)\big), \tag{40}$$

$$\alpha U = 4s_W^2 \big(\Pi'_{WW}(0) - c_W^2\Pi'_{ZZ}(0) - 2s_W c_W \Pi'_{Z\gamma}(0) - s_W^2\Pi_{\gamma\gamma}(0)\big). \tag{41}$$

The job of testing new physics then boils down to calculating these vacuum polarisations in the new theory, taking the derivative

$$\Pi'_{(XY)}(0) = \frac{d}{d(q^2)}\Big[\Pi_{(XY)}(q^2)\Big]_{q^2=0},$$

and evaluating the result against the SM values. As of the last PDG review the measured values are

$$S = 0.01 \pm 0.10(-0.08), \tag{42}$$

$$T = 0.03 \pm 0.11(+0.09), \tag{43}$$

$$U = 0.06 \pm 0.10(+0.01). \tag{44}$$

These values were calculated assuming $M_{\rm Higgs} = 117$ GeV and the corrections in parentheses show the difference if we instead assume $M_{\rm Higgs} = 300$ GeV.

### 3.3.2 Oblique operators

There are obviously a large number of dimension 6 operators we can form from the fields of the SM[18]. For any observable in our new theory, we expect that predictions for that operator will take the form

$$X_{\text{pred}} = X_{\text{SM}} + \sum_i a_i X_i,$$

where $X_{\text{SM}}$ is the prediction for that observable in the SM and the $X_i$ are the corrections introduced for each $\mathcal{O}_i$. Not all operators contribute to all observables, of course. Those that do contribute might do so in one of two ways, which we call direct and indirect. A direct correction comes from the introduction of a new channel for the process that observable derives from. So, if we are considering the process $e^+e^- \to \mu^+\mu^-$ and the observable is the cross-section, then we would get a direct contribution from a four-fermion operator like $\mathcal{O}_{el} = (\bar{\ell}\gamma^\mu \ell)(\bar{e}\gamma_\mu e)$. An indirection contribution, on the other hand, arises from a correction to processes used in measuring input parameters such as $\alpha, G_F$ and $m_Z$. Since these are used as inputs to calculate other observables, those all receive corrections from these shifts.

We will focus on the class of operators which do not contain fermion fields. In our effective theory formulation these arise when the new heavy fields couple directly to the SM gauge fields and Higgs. Since they directly correct gauge boson masses and fermion couplings, they are also referred to as 'universal' operators. The main reason for focusing on such operators is that even small couplings of new physics to the SM quarks and leptons can lead to very large (experimentally disallowed) flavour changing neutral currents. Note that new fields which couple to the SM gauge bosons can give rise to effective interactions involving the quarks and leptons. Luckily, the coefficients of these operators are suppressed both by inverse powers of $\Lambda$ and by what Grinstein calls a "weak coupling suppression factor" $\sim \alpha/4\pi$. Therefore they can be ignored when focusing on oblique operators.

Skiba identifies two operators as being particularly interesting:

$$\mathcal{O}_S = H^\dagger \sigma^i H W^i_{\mu\nu} B^{\mu\nu} \tag{45}$$

$$\mathcal{O}_T = \left| H^\dagger D_\mu H \right|^2 \tag{46}$$

They generate the vertices shown in Fig. 6. $O_T$ introduces kinetic mixing between $W^3_\mu$ and $B_\mu$, while $O_T$ violates custodial symmetry. These operators are interesting because their couplings can be directly related to the Peskin-Takeuchi parameters:

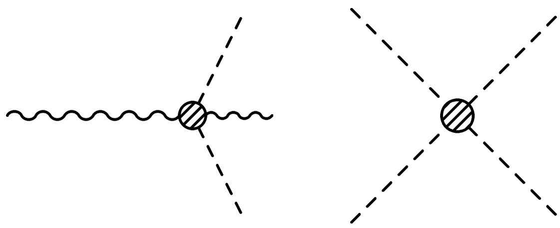
$$S = \frac{4 s_W c_W v^2}{\alpha} a_S, \tag{47}$$

$$T = -\frac{v^2}{2\alpha} a_T. \tag{48}$$

## 3.4 Constraints on new physics

Consider a new heavy quark familyPeskin & Takeuchi introduce this example in their original paper, and Skiba discusses it in his review. $(T, B)$ with the same structure as SM quark families (the left-handed portions form a doublet under $SU(2)$), hypercharge

[18]This section is largely based on Han [19] and Skiba [5], with minor input from Grinstein [20].



(a) $O_S$ (b) $O_T$

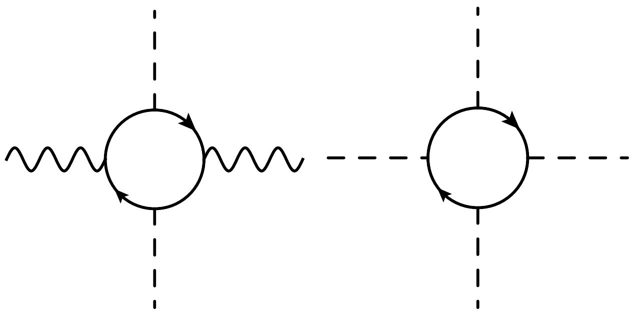
Figure 6: Vertices arising from Skiba's oblique operators. Dashed lines are the Higgs doublet, and the boson lines in (a) are $A^i_\mu$ and $B_\mu$ left to right.

$Y$ and masses $m_T, m_B$. These would generate oblique corrections when integrated out if they do not couple directly to SM fermions. We can write down a Lagrangian which satisfies these conditions easily.

$$\mathcal{L}_{\text{new}} = i\bar{Q}_L \not{\mathcal{D}} Q_L + i\bar{T}_R \not{\mathcal{D}} T_R i\bar{B}_R \not{\mathcal{D}} B_R - \left[ y_T \bar{Q}_L H T_R + y_B \bar{Q}_L H B_R + \text{H.c.} \right] \tag{49}$$

The $Q_L = (T, B)_L$ is the left-handed $SU(2)$ doublet, so the first three terms are then standard fermion kinetic terms, with $\mathcal{D}$ the SM covariant derivative (we have not changed the gauge sector). The Yukawa couplings have been labelled $y_{T,B}$ and "H.c." indicates that the Hermitian conjugates of the prior to terms in the bracket are also included.

Note that $\mathcal{L}_{\text{new}}$ is (part of) the Lagrangian for the 'full theory' and not the effective Lagrangian. We need to perform a matching calculation to include the specifics of this model into our general effective Lagrangian (28). We do this by calculating the amplitudes of the interactions represented by $\mathcal{O}_{S,T}$ in the full theory. The simplest way of producing these interactions is at the one loop level, through the diagrams shown in Fig. 7.



(a) (b)

Figure 7: 1-loop corrections to the $\mathcal{O}_S$ and $\mathcal{O}_T$ interactions. The new physics contribution comes from $T$, $B$ in the loop.

In order to calculate these diagrams, we need to turn on the Higgs background, and work with a Higgs vev. This is because chiral quarks are massless until expanded around a Higgs vev. Skiba does this, and then obtains the following expansion of $\mathcal{O}_T$:

$$\mathcal{O}_T = |H^\dagger \mathcal{D}_\mu H|^2 = \frac{v^4}{4}\frac{g_2^2}{4}(W_{3\mu})^2 + \ldots$$

where we have ignored terms in the expansion with derivatives and $B_\mu$ fields. Note that this is a mass term, which means we'll need to calculate vacuum polarisations in the full theory — corrections to the $W_{3\mu}$ propagator.

A similar expansion of $\mathcal{O}_S$ around the Higgs vev yields

$$\mathcal{O}_S = H^\dagger \sigma^i H W^i_{\mu\nu} B^{\mu\nu} = -\frac{v^2}{2} W^3_{\mu\nu} B^{\mu\nu} + \dots$$

This is kinetic mixing between the $W^3$ and $B$ gauge fields, as expected.

We would now need to perform the calculations for the full theory interactions for these processes. As stated above, these are one loop diagrams. Skiba give the following for the final contributions to $S$ and $T$:

$$S = \frac{g_1 g_2 N_c}{6(4\pi)^2}\left[1 + 2Y \ln\left(\frac{m_B^2}{m_T^2}\right)\right],$$
$$T = -\frac{2N_c}{v^2\alpha^2(4\pi)^2}\left[\frac{m_B^2 m_T^2 \ln\left(\frac{m_T^2}{m_B^2}\right) - \frac{1}{2}m_T^4 + \frac{1}{2}m_B^4}{m_T^2 - m_B^2}\right],$$

where $N_c = 3$ is the number of colours, and the other symbols have their usual meanings. The experimental bounds on $\rho$ tightly constrain the mass splitting to be very small within any new quark doublet, i.e., $\Delta m = |m_B - m_T| \ll m_B, m_T$. If we assume they are degenerate we can simply the expression for $S$ to

$$S \approx \frac{g_1 g_2 N_c}{6(4\pi)^2}. \tag{50}$$

We can calculate very roughly what this means. Taking $m_Z = 91.18$Gev, $m_W = 80.38$Gev, $v = 246$Gev, we can work out $g_1, g_2$ from equations (33) and (34). Plugging all of these into equation (50) yields

$$S \approx 0.0007241.$$

This is within experimental error of the current best fit from the PEW data to $S$. The nature of these results is of course that exclusions are far more useful than allowances. This says nothing of the likelihood of a degenerate fourth generation of quarks, it simply says it is not disallowed by this test.

Peskin and Takeuchi performed another comparison which is more decisive. They analyse a technicolour model and find that $S$ is proportional to the number of techni-fermion doublets. Hewett, discussing their paper, quotes the size of $S$ with one full techni-fermion generation as $S \approx 1.62$, which is clearly excluded by the PEW tests.

We therefore have a relatively simple way of testing new physical models. If their additions to the $S, T$, or $U$ parameters falls outside the PEW data bounds, they cannot be true physical theories.

# 4 Conclusion

Effective field theories are powerful tools in physics, particularly in high energy theory. Given a known UV theory, they allow us to specialise to simpler low energy theories when working in low energy domains. In section 2 we began with a general effective Lagrangian containing an infinite sum of local operators, constructed out of the light fields still in the theory and obeying the symmetries of the full theory. We then saw how to apply power counting arguments to truncate this sum to the most significant operators. The coefficients of these are then found by matching a calculation of a specific observable in both the full and effective theories. In this way we derive a theory which can calculate observables for all processes which involve just these light fields and which occur within the EFT's energy range. This simplifies calculations, and can be pedagogically useful. In addition to this, important insights into the structure of quantum field theory have come from the study of EFTs, particularly regarding renormalisability as a criterion for 'good' theories. We learn by studying EFTs that non-renormalisable interactions are acceptable as long as we cast our theory as an EFT, valid up to a cut-off.

Effective field theories also provide a useful and conceptually simple way of testing models of physics beyond the Standard Model. In section 3 we constructed a general effective theory containing the SM and a set of local operators. This can be matched to any model of new physics, provided the new fields decouple in the large $\Lambda$ limit. These methods are most adept at testing models of new physics which generate "oblique" corrections to the SM. We can constrain the effects of the general local operators by comparison to the precision electroweak data. Rather than computing the effects of each new model on current observables, we relate these models to the pre-defined parameters $S, T$ and $U$, which have already been constrained. This is done by determining how the new physics contributes to oblique operators with known relations to $S, T$ and $U$.

# References


[1] J. Polchinski, "Effective Field Theory and the Fermi Surface," `arxiv:hep-th/9210046v2`.

[2] C. Burgess, "Lecture 2a: Effective field theories (PIRSA09090017)." Video lecture, 2009. `http://pirsa.org/C09020`. PSI.

[3] C. Burgess, "Lecture 3a: Effective field theories (PIRSA09100050)." Video lecture, 2009. `http://pirsa.org/C09020`. PSI.

[4] C. Burgess, "Lecture 4b: Effective field theories (PIRSA09100154)." Video lecture, 2009. `http://pirsa.org/C09020`. PSI.

[5] W. Skiba, "TASI lectures on Effective Field Theory and Precision Electroweak Measurements," `arxiv:hep-ph/1006.2142v1`.

[6] J. Bain, "Effective Field Theories," `http://ls.poly.edu/ jbain/papers/EFTs.pdf`.

[7] D. Kaplan, "Five lectures on Effective Field Theory," `arxiv:nucl-th/0510023v1`.

[8] T. Cao and S. Schweber, "The Conceptual Foundations and the Philosophical Aspects of Renormalization Theory," *Synthese* **97** (1993) 33–108.

[9] H. Georgi, "Effective Field Theory," *Annual Review of Nuclear and Particle Science* **43** (1993) 209–252.

[10] H. Georgi, "Thoughts on Effective Field Theory," *Nuclear Physics B (Proc. Suppl.)* **29B** (1992) 1–10.

[11] A. Pich, "Effective Field Theory," `arxiv:hep-ph/9806303`.

[12] D. Kaplan, "Effective Field Theories," arxiv:nucl-th/9506035v1.

[13] J. de Blas Mateo, *Effective Lagrangian Description of Physics Beyond the Standard Model and Electroweak Precision Tests*. PhD thesis, University of Granada, 2010. http://www-ftae.ugr.es/files/PhDThesis_JdeBlas.pdf.

[14] H. Osborn, "Part III lecture notes on the Standard Model," damtp.cam.ac.uk/user/ho/SM.ps.

[15] L. Ryder, *Quantum Field Theory*. CUP, 2nd ed., 1996.

[16] The ALEPH, DELPHI, L3, OPAL, SLD Collaborations, the LEP Electroweak Working Group, the SLD Electroweak and Heavy Flavour Groups, "Precision Electroweak Measurements on the Z Resonance," arxiv:hep-ex/0509008.

[17] K. Nakamura *et al.*, "The Review of Particle Physics," *Journal of Physics* **G37** (2010) 075021.

[18] J. Hewett, "The Standard Model and why we believe it," arxiv:hep-ph/9810316v1.

[19] Z. Han and W. Skiba, "Effective Theory Analysis of Precision Electroweak Data," arxiv:hep-ph/0412166.

[20] B. Grinstein and M. Wise, "Operator analysis for precision electroweak physics," *Physics Letters B* **265** (1991) 326–34.