Functions – supplement for PH104

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A mapping, or function, is a rule that associates elements of one set with elements of another set. If X and Y are sets, and f is a function from X to Y, then we write $f : X \to Y$. f is sometimes also called a *mapping*, the set X is called the *domain*, and Y is called the *codomain*.

We specify what the rule is by writing $f(x) = y$. e.g. $X = \{1, 2, 3\}, Y = \{2, 4, 6\}$, the map $f(x) = 2x$ associates each element $x \in X$ with the element in Y that is double it.

We sometimes refer to $f(X)$, which is called the *image* of X under f. This is the set of points in Y that f maps onto. E.g., In figure 1 below, $f(X) = \{A, B, D\}$.

Functions and logical relations. Functions are a special kind of *relation*, which you worked with extensively in the first-order logic section. The new notation $f(x) = y$ is requivalent to writing fxy. The relations we worked with there could hold between one object and any number of others (think of the relation "greater than", which holds between the number 1 and infinitely many numbers). In order to count as a function, a relation must associate *just one* element in the codomain with each element in the domain. In other words, it cannot be a one-to-many relation like "greater than".

Formal definition. A function from a set X to set Y, denoted $f : X \to Y$, is a relation between all the elements of X and elements of Y (possibly not all), that assigns a unique element $y \in Y$ to every $x \in X$. We denote this $y = f(x)$.

$$\forall x \in X, \exists y \in Y, f(x) = y \wedge \forall z \in Y \big(f(x) = z \to z = y\big)$$

Injective (one-to-one) function: A function is injective if it takes each element of the domain onto at most one element of the codomain. It never maps more than one element in the domain onto the same element in the codomain. Formally, if f is a function between set X and set Y, then f is injective iff

$$\forall a, b \in X, f(a) = f(b) \to a = b$$

Surjective (onto) function: A function is surjective if it maps something onto every element of the codomain. It can map more than one thing onto the same element in the codomain, but it needs to 'hit' everything in the codomain. Formally, if f is a function between set X and set Y, then f is surjective iff

$$\forall y \in Y, \exists x \in X, f(x) = y$$

Bijective function (one-to-one correspondence): A function is bijective if it is both injective and surjective; or in the alternate vocabulary, both one-to-one and onto.

[1]Images from Wikipedia.

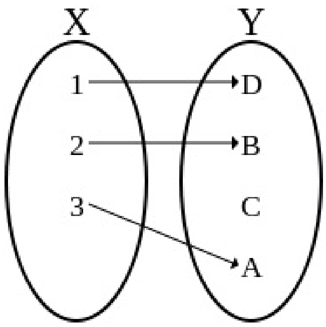

Figure 1: Injective map.

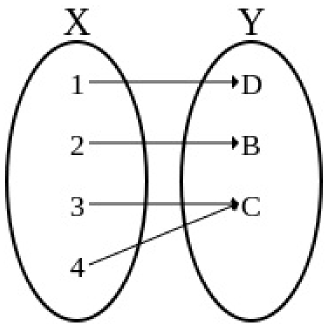

Figure 2: Surjective map.

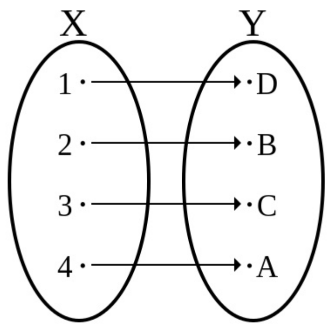

Figure 3: Bijective map.

Recall from the lectures that two sets are called "equinumerous" just when there is a bijection between them.

Composition of functions. Suppose we have three sets, X, Y, Z, and $f : X \to Y$, $g : Y \to Z$ are functions. We can form a new function, mapping from X to Z by *composing* the functions f and g, to create a new function which maps $x \in X$ to $g(f(x)) \in Z$. The resulting *composite* function is denoted $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.[2]

Properties of composition:

1. **The composition of two injections is an injection.** Let $f : A \to B$, and $g : B \to C$ be injective functions. Let $a_1, a_2 \in A$ and suppose that $g(f(a_1)) = g(f(a_2))$. Because g is injective, we know that $f(a_1) = f(a_2)$ –see the definition of injective above. But f is also injective, so $a_1 = a_2$. So $g(f(a_1)) = g(f(a_2)) \to a_1 = a_2$, which makes $g \circ f$ injective.

2. **The composition of two surjections is a surjection.** Let $f : A \to B$, and $g : B \to C$ be surjective functions. To show $gcircf$ is a surjection we need to take any $c \in C$ and find an $a \in A$ that $g \circ f$ maps to c. For any $c \in C$, we know there exists $b \in B$ such that $g(b) = c$, because g is a surjection. Now because f is a surjection, for that $b \in B$, there exists $a \in A$ such that $f(a) = b$. So $(g \circ f)(a) = g(f(a)) = g(b) = c$, and $g \circ f$ is a surjection.

3. **The composition of two bijections is a bijection.** This follows from (1) and (2) above.

[2]Frustratingly, there are two conventions for the notation of composite functions. Here I use $(g \circ f)(x) = g(f(x))$. The alternate notation uses $(f \circ g)(x) = g(f(x))$. Be careful if consulting another source!